Convergence of a sequence

Instructor's Notes

Introduction

This applet illustrates the ϵ-M definition of the convergence a sequence encountered in real analysis courses. Recall that a sequence $f : \mathbb{N} \to \mathbb{R}$ converges to a real number L as $n \to \infty$ if, for all $\epsilon > 0$, there is some natural number M such that for all $n > M$, the point $f_n := f(n)$ is less than ϵ away from L, i.e. $|f_n - L| < \epsilon$. This is written $\lim_{n\to\infty} f_n = L$.

About the applet

General description and functionalities

In the applet, all of these parameters are controllable and visualised. You can enter a rule for f_n and set a candidate limit L. You can toggle ϵ and M on and off, determining the ϵ "envelope" around L and the ''cutoff" M. When both ϵ and M are on, "bad" points which satisfy $n > M$ but do not lie within ϵ of L are labelled red. "Good" points which satisfy $n > M$ and $|f_n - L| < \epsilon$ are labelled green. "Irrelevant" points with $n \leq M$ are gray. You can also add a second sequence to illustrate bounding arguments. The formal definition is at the bottom of the page and colour coded to match the interactive applet elements.

Notation and colour coding

The notation and colour coding in the applet have been designed to reinforce each other. The

$$\forall \varepsilon > 0 \ \exists M \in \mathbb{N} \ \forall n > M \ |f_n - L| < \varepsilon$$

Figure 1: Notation

formal definition of ϵ-M convergence is displayed at the bottom of the applet, with different parts coloured to match the corresponding applet elements. The notation is consistent with courses run by the School of Mathematics and Statistics at the University of Melbourne.

Instructions for using the applet

Enter a rule to generate your sequence in the box labelled `fn`. There should be a single independent variable `n`. Geogebra recognises elementary functions like `sin`, `cos`, `sqrt`, and `ln`. A complete list can be found here.

Drag the brown circle ●, or the connected horizontal line, to change the putative limit L. The ϵ envelope around L can be toggled on and off. Once on, it can be controlled by dragging the orange circle ● on the vertical axis or moving the horizontal boundaries. The cutoff M can also be toggled. The region to the right of M is blue and can be changed by dragging the blue circle ● or the attached vertical line. With ϵ and M both on,

- "irrelevant" points in the sequence f_n which do not lie in the M region are gray;
- "bad" points which lie in the M region but not the ϵ envelope are red; and
- "good" points which lie in both the M region and the ϵ envelope are green.

Add another sequence by toggling the `Show second sequence` button. A new input box labelled gn will appear. The points in this sequence are labelled with diamonds ◆. If ϵ and M are both on, points of g_n are coloured according to the same scheme as f_n.

Example usage

Here are some examples illustrating key concepts which have been tested on the applet.

1. Enter the sequence $f_n = 1/n$ and set $L = 0$. If you decrease ϵ, you must increase M to ensure that there are no red points. A very short proof shows that $M > \epsilon^{-1}$ will always work, and this can be verified with the applet. This is a simple example of convergence, and also shows how the choice of cutoff M depends on ϵ.

2. In the example above, set $L = 1$ and $\epsilon = 0.5$. No matter the choice of cutoff M, there will always be red points. This is an example of a sequence *not* converging to L.

3. Enter $f_n = (-1)^n$. Set L to any value desired and $\epsilon = 0.5$. It should be clear that no choice of M will eliminate red points. Since this is true for any value of L, we have a simple example of a divergent sequence.

4. We give a “real-world” example of bounding. Enter $f_n = 3n/(2n^2 - 1)$ and set $L = 0$. Toggle the second sequence and enter $g_n = 3/n$. Turn on ϵ and M. As students can prove analytically, it is possible to eliminate red points of g_n by choosing $M \geq 3/\epsilon$ for any ϵ. As the applet suggests visually, and a quick calculation confirms, points of f_n are trapped between the points of g_n and 0. Hence, we can also eliminate red points of f_n with the same $M \geq 3/\epsilon$. It follows that f_n also converges to $L = 0$. This illustrates how we can bound one sequence with another to ensure convergence.

Suggestions for using the applet

The applet can be used to illustrate

- the dependence of the cutoff M on the envelope width (example 1);
- why, in terms of ϵ-M, a sequence converges to a limit L (example 1);
- why, in terms of ϵ-M, a sequence does not converge to L (example 2);
- why, in terms of ϵ-M, a sequence does not converge at all (example 3);
- how bounding with a simpler sequence can indicate convergence (example 4).

Students can use the applet to 'sanity check' their own bounding arguments during the 'pre-proof' stage of their convergence proofs. Some possible classroom uses for the applet:

- as a lecture demonstration to accompany the formal definition;
- in computer labs, where students can explore the definition interactively;
- in assignments, or as part of computer-based assessment;
- for students to use outside of class as a learning tool;
- in one-on-one consultation.